EQUIVARIANT COHOMOLOGY OF CERTAIN MODULI OF WEIGHTED POINTED RATIONAL CURVES

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1. Introduction

In [8] Hassett introduces and studies the moduli spaces of weighted pointed stable curves. A weighted pointed curve is a nodal curve with a sequence of smooth marked points, each assigned a rational number between 0 and 1. A subset of the marked points may coincide if the sum of their weights is at most 1.

The moduli spaces are connected, smooth and proper Deligne-Mumford stacks. In the special case of genus zero the moduli spaces are smooth projective varieties. Throughout this paper we work over $\mathbb{C}$ as the base field and we always consider cohomology with $\mathbb{C}$ coefficients.

Consider the weight data

$$\mathcal{A}(m,n) = \Bigg(\underbrace{1,\dots,1}_{m},\underbrace{1/n,\dots,1/n}_{n}\Bigg) \quad m+n\geq 3,\ m\geq 2\ . \tag{1}$$

Let

$$\overline{M}_{0,m|n} = \overline{M}_{0,\mathcal{A}(m,n)}\ .$$

$\overline{M}_{0,m|n}$ parametrises nodal curves with $m+n$ smooth marked points such that the first m marked points are distinct but any subset of the last n marked points can coincide. There is naturally an action of $S_m \times S_n$ on $\overline{M}_{0,m|n}$. Here S_m permutes the first m marked points and S_n permutes the last n.

In this paper we study the induced action of $S_m \times S_n$ on the cohomology of $\overline{M}_{0,m|n}$ and calculate the equivariant Poincaré polynomial for some small values of m and n.

Let $M_{0,m|n}$ be the interior of the the moduli space, parametrizing only the smooth curves. We first derive the $S_m \times S_n$ character on $H^*(M_{0,m|n})$, and write down a generating function for the characters. We then describe a recipe for calculating the generating function for the $S_m \times S_n$ character of $H^*(\overline{M}_{0,m|n})$. This is achieved by analysing a spectral sequence relating the cohomology of $M_{0,m|n}$ to that of $\overline{M}_{0,m|n}$.

It should be noted that when $n = 0$, we simply get the moduli of stable rational curves with m marked points. In this case the equivariant cohomology was studied by Getzler [6].

In another direction when $m = 2$, the moduli spaces under consideration are the Losev- Manin spaces of [9]. The $S_2 \times S_n$ action on the cohomology was determined in this case by Bergström and Minabe [3].

Finally Bergstrom and Minabe [2] give a recursive method for calculating the equivariant Poincaré polynomial of $\overline{M}_{0,m|n}$ for all m and n. However our method seems more direct. We use techniques developed by Getzler [6] and Getzler and Kapranov [7]. We adopt the notation $\overline{M}_{0,m|n}$ from [3].

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2. Preliminaries on $\overline{M}_{0,m|n}$

Let $\mathcal{A}(m,n)$ be as in (1). Following Hassett [8], $\overline{M}_{0,m|n}$ is the moduli of weighted pointed stable curves of of genus zero corresponding to the weight data $\mathcal{A}(m,n)$.

When $m > 3$, $\overline{M}_{0,m|0}$ is simply the moduli of stable rational curves with m marked points. We abbreviate it as $\overline{M}_{0,m}$.

Denote by $M_{0,m|n}$ the open subvariety parametrising the smooth curves.

2.1. **The stable curves.** An $\mathcal{A}(m,n)$-stable curve $(C; p_1, \dots, p_{m+n})$ is a nodal curve with smooth marked points p_i. The marked points of C along with the nodes will be called special points. We shall call the first m marked points along with the nodes special points of type 1, whereas the last n marked points will be referred to as special points of type 2. The curve C satisfies the following,

- Arithmetic genus of C is 0.
- The points $\{p_1, \dots, p_m\}$ are all distinct.
- Any subset of the points $\{p_{m+1}, \dots, p_{m+n}\}$ can coincide, but these points are all distinct from $\{p_1, \dots, p_m\}$.
- Any irreducible component of C has at least 3 special points with at least 2 of type 1.

The varieties $\overline{M}_{0,m|n}$ are smooth and projective and $\overline{M}_{0,m|n} \backslash M_{0,m|n}$ is a divisor with normal crossings. Ceyhan [4] studies the cohomology of $\overline{M}_{0,\mathbf{A}}$, for any weight data $\mathbf{A}$. As a special case it follows that all the cohomology of $\overline{M}_{0,m|n}$ is algebraic. This means that all the odd degree cohomology groups vanish and the even cohomology groups are isomorphic to the Chow groups

$$H^{2i+1}(\overline{M}_{0,m|n}, \mathbb{Q}) = 0 \quad \text{and} \quad H^{2i}(\overline{M}_{0,m|n}, \mathbb{Q}) \cong A^i(\overline{M}_{0,m|n}, \mathbb{Q}).$$

2.2. **Dual graphs.** A graph will be a triple (F, V, σ). Where

(1) F is the set of flags;
(2) V is a partition of F;
(3) σ is an involution on F.

The parts of V are the vertices of the graph. For $v \in V$, let $F(v) = \{f \in v \mid f \in F\}$ be the flags incident on v. The fixed points of σ are the leaves. The set of leaves will be denoted by L, and those incident on a vertex v denoted as $L(v)$. The two cycles of σ will be the edges of the graph and the set of edges denoted by E.

Colouring of a graph G consists of a set X and a function $c : F(G) \to X$ such that $c(f) = c(\sigma f)$ for every flag f. A colouring assigns a colour (an element of X) to each flag such that both flags of an edge have the same colour. It thus makes sense to talk about the colour of an edge.

Geometric realisation of a graph G, denoted by $|G|$ is a topological space. It is the quotient space of, the collection of intervals indexed by the flags of G, by an equivalence relation.

$$|G| = \frac{F(G) \times [0,1]}{\sim}$$

Here $(f_1, 0) \sim (f_2, 0)$ if the flags f_1, f_2 are incident on the same vertex and $(f, 1) \sim (f', 1)$ if the flags f, f' are part of an edge.

A tree T is a graph such that $|T|$ is connected and simply connected.

The dual graph of an $\mathcal{A}(m,n)$-stable curve is a tree coloured by $\{1, 2\}$. The tree has one flag for each marked point and two for each node. For every irreducible component it has a vertex. The marked points correspond to the leaves and the nodes correspond to the edges. The flags corresponding to the special points of type 1 have colour 1 where as the flags corresponding to the special points of type 2 have colour 2. Further the leaves are numbered 1 to $m+n$ according to the marked point it represents.

2.3. **Strata of $\overline{M}_{0,m|n}$.** It is clear that the dual graphs of $\mathcal{A}(m,n)$-stable curves have to satisfy certain constraints. Let T be such a dual graph. For any vertex $v \in V(T)$ let $F_1(v)$ be the flags of colour 1 and $F_2(v)$ the flags of colour 2. Then we must have $|F(v)| \geq 3$ and $|F_1(v)| \geq 2$. Let us call such trees $\mathcal{A}(m,n)$-stable and denote the isomorphism classes of such trees by $\mathbb{T}(m,n)$.

For any $T \in \mathbb{T}(m,n)$ let $M(T)$ be the subvariety of $\overline{M}_{0,m|n}$ parametrising curves whose dual graphs are isomorphic to T. Let $\overline{M}(T)$ be the closure. It is clear that (see Ceyhan [4, Section 3])

$$M(T) \cong \prod_{v\in V(T)} M_{0,\#F_1(v)|\#F_2(v)} \quad \text{and} \quad \overline{M}(T) \cong \prod_{v\in V(T)} \overline{M}_{0,\#F_1(v)|\#F_2(v)}.$$

The codimension of $\overline{M}(T)$ is equal to the number of edges $|E(T)|$ of T.

We have a stratification by dual graphs

$$\overline{M}_{0,m|n} = \coprod_{T\in\mathbb{T}(m,n)} M(T).$$

3. Symmetric Group Representations

3.1. **Symmetric functions.** . For results and notation of this section we refer to Macdonald [10]. Let $\Lambda = \lim\limits_{\leftarrow} \mathbb{Z}[\![x_1,\ldots,x_n]\!]^{S_n}$ be the ring of symmetric functions. It is well known that

$$\Lambda \otimes \mathbb{Q} = \mathbb{Q}[\![p_1,p_2,\ldots]\!]$$

where $p_k = \sum_{i=1}^{\infty} x_i^k$ are the power sums. Let $\lambda = (\lambda_1,\ldots,\lambda_k)$ be a partition of n, which we denote by $\lambda \vdash n$; define $p_\lambda = p_{\lambda_1}\cdots p_{\lambda_k}$. For an S_n representation V we define the symmetric function

$$\mathrm{ch}_n(V) = \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{Tr}_V(\sigma) p_{\lambda(\sigma)} \ ,$$

here $\lambda(\sigma)$ is the partition corresponding to the cycle decomposition of σ.

The irreducible representations of S_n are indexed by partitions of n. For $\lambda \vdash n$ let V_λ be the corresponding irreducible representation. The Schur functions also indexed by partitions of n are defined as

$$s_\lambda = \mathrm{ch}_n(V_\lambda).$$

Schur functions $\{s_\lambda \mid \lambda \vdash n, n \geq 1\}$ form an additive basis of Λ. There are also the elementary symmetric functions $e_n = s_{1^n}$ and the complete symmetric functions $h_n = s_n$.

There is an associative product $\circ$ on Λ called plethysm. It is characterised by the fact that

$$\mathrm{ch}_n\left(\mathrm{Ind}_{S_k\wr S_n}^{S_{kn}} V_1 \boxtimes V_2 \boxtimes \cdots \boxtimes V_2\right) = \mathrm{ch}_k(V_1) \circ \mathrm{ch}_n(V_2) \ ,$$

where $S_k \wr S_n$ is the wreath product $S_k \ltimes (S_n)^k$, V_1 is a representation of S_k and V_2 is a representation of S_n.

Let $\Lambda^{(2)} = \Lambda \otimes \Lambda$. We denote the symmetric functions in the first tensor factor by the superscript (1) and those in the second tensor factor by the superscript (2).

For V a representation of $S_m \times S_n$ we define

$$\mathrm{ch}_{m|n}(V) = \frac{1}{m! \times n!}\sum_{(\sigma,\tau)\in S_m\times S_n} \mathrm{Tr}_V(\sigma,\tau) p^{(1)}_{\lambda(\sigma)} p^{(2)}_{\lambda(\tau)} \in \Lambda^{(2)}.$$

We shall need the following result later on.

Proposition 3.1. *Let W be any representation of S_n and $\mathbf{D}$ the following differential operator on Λ*

$$\mathbf{D} = p_1 \frac{\partial}{\partial p_1} - 1 \ ,$$

then

$$\mathrm{ch}_n\left(W \otimes V_{(n-1,1)}\right) = \mathbf{D}\,\mathrm{ch}_n(W) \ .$$

$V_{(n-1,1)}$ is the irreducible representation corresponding to the partition $(n-1,1)$ and often referred to as the standard representation of S_n.

Proof. Let $\operatorname{fix}(\sigma)$ denote the number of fixed points of $\sigma \in S_n$. Recall that $\operatorname{Tr}_{V_{(n-1,1)}}(\sigma) = \operatorname{fix}(\sigma) - 1$. Also note that $\lambda(\sigma) = (1^{\operatorname{fix}(\sigma)}, 2^{a_2}, \ldots)$; so $p_{\lambda(\sigma)} = p_1^{\operatorname{fix}(\sigma)} p_2^{a_2} \cdots p_n^{a_n}$. Thus

$$p_1 \frac{\partial p_{\lambda(\sigma)}}{\partial p_1} = \operatorname{fix}(\sigma) p_{\lambda(\sigma)} \ .$$

Hence

$$\begin{aligned}
\operatorname{ch}_n\left(W \otimes V_{(n-1,1)}\right) &= \frac{1}{n!} \sum_{\sigma \in S_n} (\operatorname{fix}(\sigma) - 1) \operatorname{Tr}_W(\sigma) p_{\lambda(\sigma)} \\
&= \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{Tr}_W(\sigma) \left(p_1 \frac{\partial p_{\lambda(\sigma)}}{\partial p_1} - p_{\lambda(\sigma)} \right) = p_1 \frac{\partial \operatorname{ch}_n(W)}{\partial p_1} - \operatorname{ch}_n(W) \ .
\end{aligned}$$

□

3.2. $\mathbb{S}$ **modules.** An $\mathbb{S}$ module (as in [6, §1]) $\mathcal{V}$ is a sequence of graded vector spaces $\{\mathcal{V}(n) \mid n \in \mathbb{N}\}$ with an action of S_n on $\mathcal{V}(n)$. The characteristic of an $\mathbb{S}$ module is defined as a symmetric series in $\Lambda[\![t]\!]$

$$\operatorname{ch}_t(\mathcal{V}) = \sum_{n=1}^{\infty} \sum_{i \in \mathbb{Z}} (-t)^i \operatorname{ch}_n(\mathcal{V}^i(n)).$$

Here $\mathcal{V}^i(n)$ is the i-th graded component of $\mathcal{V}(n)$.

Similarly an $\mathbb{S}^2$ module $\mathcal{W}$ is a collection of graded vector spaces $\{\mathcal{W}(m,n) \mid (m,n) \in \mathbb{N}^2\}$, with an action of $S_m \times S_n$ on $\mathcal{W}(m,n)$. We define the characteristic in an analogous way

$$\operatorname{ch}_t(\mathcal{W}) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{i \in \mathbb{Z}} (-t)^i \operatorname{ch}_{m|n}(\mathcal{W}^i(m,n)) \in \Lambda^{(2)}[\![t]\!] \ .$$

In the case of an ungraded $\mathbb{S}^2$ module $\mathcal{W}$ we write the characteristic as $\operatorname{ch}(\mathcal{W})$. We define the $\mathbb{S}^2$ module $\mathbb{T}\mathcal{W}$ in the following way

$$\mathbb{T}\mathcal{W}(m,n) = \bigoplus_{T \in \mathbb{T}(m,n)} \mathcal{W}(T) \ .$$

Here $\mathbb{T}(m,n)$ are the isomorphism classes of $\mathcal{A}(m,n)$ stable trees and

$$\mathcal{W}(T) = \bigotimes_{v \in V(T)} \mathcal{W}(F(v)) \ .$$

For a more detailed discussion see Getzler and Kapranov [7].

3.3. **Partial Legendre transform.** Let $\operatorname{rk} : \Lambda^{(2)} \to \mathbb{Q}[\![x,y]\!]$ be the homomorphism such that $\operatorname{rk}\left(p_1^{(1)}\right) = x$, $\operatorname{rk}\left(p_1^{(2)}\right) = y$ and $\operatorname{rk}\left(p_n^{(i)}\right) = 0$ for $n > 1$ and $i = 1, 2$. Thus if V is a representation of $S_n \times S_m$ then

$$\operatorname{rk}(\operatorname{ch}_{m|n}(V)) = \frac{\dim V}{m!n!} x^m y^n \ .$$

Let $\mathbb{Q}[\![x,y]\!]_*$ be the power series of the form $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} x^i y^j$ where $a_{2,0} \neq 0$ and $a_{i,j} = 0$ if $i < 2$ or $i + j < 3$. Let $\Lambda_*^{(2)} = \operatorname{rk}^{-1} \mathbb{Q}[\![x,y]\!]_*$.

To define the partial Legendre transform we first define a variant of plethysm; $\circ_{(1)}$ which is an associative product on $\Lambda^{(2)}$:

(1) $f \mapsto f \circ_{(1)} g$ is a homomorphism $\Lambda^{(2)} \to \Lambda^{(2)}$, for any $g \in \Lambda^{(2)}$
(2) $g \mapsto p_n^{(i)} \circ_{(1)} g$ is a homomorphism $\Lambda^{(2)} \to \Lambda^{(2)}$,
(3) $p_n^{(1)} \circ_{(1)} p_k^{(i)} = p_{nk}^{(i)}$ and $p_n^{(2)} \circ_{(1)} p_k^{(i)} = p_n^{(2)}$.

For $f \in \Lambda_*^{(2)}$ there is a unique $g \in \Lambda_*^{(2)}$ satisfying the equation

$$g \circ_{(1)} \frac{\partial f}{\partial p_1^{(1)}} + f = p_1^{(1)} \frac{\partial f}{\partial p_1^{(1)}} . \tag{2}$$

For the existence and uniqueness we refer to Getzler and Kapranov [7, Theorem 7.15]. The proof in this case is completely analogous and goes through in almost the same way without any subtleties. We call the function g the partial Legendre transform of f and denote it by $\mathfrak{L}^{(1)}f$.

A little bit of algebra shows that $\mathfrak{L}^{(1)}$ is an involution on $\Lambda_*^{(2)}$, that is $\mathfrak{L}^{(1)}\mathfrak{L}^{(1)}f = f$. We have the following result.

Proposition 3.2. *Let $\mathcal{W}$ be an ungraded $\mathbb{S}^2$ module such that $\mathcal{W}(m,n) = 0$ if $m < 2$ or $m+n < 3$. $F = e_2^{(1)} - \mathrm{ch}(\mathcal{W})$ and $G = h_2^{(1)} + \mathrm{ch}(\mathbb{T}\mathcal{W})$ are elements of $\Lambda_*^{(2)}$ and $G = \mathfrak{L}^{(1)}F$.*

The proof of this proposition is essentially the same as the proof of Theorem 7.17 of [7]. One can also look at Theorem 5.8 of [5] for a different proof.

4. Cohomology of the Interior

In this section we study the cohomology of the interior $M_{0,m|n}$. It is easy to see that

$$M_{0,m|n} \cong \left(\left(\mathbb{P}^1\right)^{m+n} \setminus \left(\bigcup_{i=1}^{m} \bigcup_{j=i+1}^{m+n} \Delta_{i,j} \right) \right) / \mathbf{PGL}(2,\mathbb{C}) , \tag{3}$$

where $\mathbf{PGL}(2,\mathbb{C})$ acts diagonally and $\Delta_{i,j} = \{(z_1,\dots,z_{m+n}) \mid z_i = z_j\}$.

Proposition 4.1. *When $m \geq 3$,*

$$H^*(M_{0,m|n}) \cong H^*(M_{0,m}) \otimes H^*(P_m^n)$$

where $P_m = \mathbb{P}^1 \backslash \{1,\dots,m\}$ is the m punctured projective plane. Moreover this decomposition respects the action of $S_m \times S_n$.

The mixed Hodge structure on $H^i(M_{0,m|n})$ is pure of weight $2i$.

Proof. Consider the fiber bundle $M_{0,m|(n+1)} \to M_{0,m|n}$ with fiber P_m. P_m is homotopic to a wedge of circles, hence a one dimensional C-W complex. The fundamental group of the base acts trivially on the fibers, (see Arnold [1]), hence in the Leray spectral sequence associated to the fibration we have

$$E_2^{p,q} \cong H^p(M_{0,m|n}) \otimes H^q(P_m).$$

Moreover the fiber bundle has a section given by

$$z_{m+n+1} = \frac{z_1 + \ldots + z_m}{m} + 2\Big(\max_{1 \leq k,l \leq m} |z_k - z_l| \Big) + 1.$$

It then follows that the only possible higher differential d_2 is trivial and we have $H^*(M_{0,m|(n+1)} \cong H^*(M_{0,m|n}) \otimes H^*(P_m)$. This completes the proof by induction on n.

The statement on the mixed Hodge structure of $H^i(M_{0,m|n})$ follows from the fact that $M_{0,m|n}$ is isomorphic to a complement of hyperplanes in a projective spaces. This can be seen from description (3). □

From the previous proposition it follows that the Poincaré polynomial of $M_{0,m|n}$, $\mathcal{P}_{M_{0,m|n}}(t)$, is the product of the Poincaré polynomials of $M_{0,m}$ and P_m^n. From Getzler [6, Section 5.6] we know that $\mathcal{P}_{M_{0,m}}(t) = (1-2t)(1-3t)\cdots(1-(m-2)t)$. It is easy to see that $\mathcal{P}_{P_m^n}(t) = (1-(m-1)t)^n$. Thus

$$\mathcal{P}_{M_{0,m|n}}(t) = \big(1-(m-1)t\big)^n \prod_{k=2}^{m-2} (1-kt) .$$

Proposition 4.1 gives a clear description of the $H^*(M_{0,m|n})$ as a representation of $S_m \times S_n$.

First note that Getzler [6] determines completely the action of S_m on $H^*(M_{0,m})$, whereas S_n acts on it trivially.

Let C_n be the vector space generated by the letters $\{x_1, \dots, x_n\}$. S_n acts on it by permuting the letters. C_n is the direct sum of the standard representation and the trivial representation i.e. $C_n = V_{(n-1,1)} \oplus V_{(n)}$.

Proposition 4.2. *We have the following description of the $S_m \times S_n$ action on the cohomology of P^n_m.*

$$
\begin{aligned}
H^k(P^n_m) &\cong \left(\otimes^k V_{(m-1,1)}\right) \boxtimes (\wedge^k C_n) \\
&\cong \begin{cases} V_{(m)} \boxtimes V_{(n)} \ , & k = 0 \\ \left(\otimes^k V_{(m-1,1)}\right) \boxtimes \left(V_{(n-k,1^k)} \oplus V_{(n-k+1,1^{k-1})}\right) \ , & 0 < k < n \\ \left(\otimes^n V_{(m-1,1)}\right) \boxtimes V_{(1^n)} \ , & k = n. \end{cases}
\end{aligned}
$$

Proof. S_m acts on P_m by permuting the punctures, so $H^1(P_m)$ is the standard representation $V_{m-1,1}$.

On the other hand S_n acts on P^n_m by permuting the factors, thus $H^1(P^n_m) \cong V_{m-1,1} \boxtimes C_n$ as a representation of $S_m \times S_n$. For $k > 1$, $H^k(P^n_m) \cong \left(\otimes^k H^1(P_m)\right) \boxtimes \left(\wedge^k C_n\right)$.

The second isomorphism follows form the decomposition $C_n = V_{(n-1,1)} \oplus V_{(n)}$. Thus for $k < n$ we have $\wedge^k C_n = \left(\wedge^k V_{(n-1,1)}\right) \oplus \left(V_{(n)} \otimes \wedge^{k-1} V_{(n-1,1)}\right)$ and $\wedge^n C_n = V_{(1^n)}$. Finally it is a fact that $\wedge^k V_{(n-1,1)} = V_{(n-k,1^k)}$. □

The Propositions 4.1 and 4.2 together give us the following decomposition for $m \geq 3$,

$$
\begin{aligned}
H^k(M_{0,m|n}) &= \bigoplus_{l=0}^{k} H^{k-l}(M_{0,m}) \otimes H^l(P^n_m) \\
&= \bigoplus_{l=0}^{k} \left(H^{k-l}(M_{0,m}) \otimes \left(\otimes^l V_{(m-1,1)}\right)\right) \boxtimes \left(\wedge^l C_n\right) \ . \qquad (4)
\end{aligned}
$$

In (4) we treat $H^{k-l}(M_{0,m})$ as just a representation of S_m.

Hence we have the following relation for $m \geq 3$

$$
\begin{aligned}
H^*(M_{0,m|n}) &= \bigoplus_{l=0}^{n} \left(H^*(M_{0,m}) \otimes H^l(P^n_m)\right) \\
&= \bigoplus_{l=0}^{n} \left(H^*(M_{0,m}) \otimes \left(\otimes^l V_{(m-1,1)}\right)\right) \boxtimes \left(\wedge^l C_n\right). \qquad (5)
\end{aligned}
$$

Let $\mathcal{M}$ be the $\mathbb{S}$ module

$$
\mathcal{M}(n) = \begin{cases} H^*(M_{0,n}) & n \geq 3, \\ 0 & n < 3. \end{cases} \qquad (6)
$$

Let $\mathfrak{m}_n = \mathrm{ch}_t(\mathcal{M}(n))$ and $\mathfrak{m} = \sum_{n=1}^{\infty} \mathfrak{m}_n = \mathrm{ch}_t(\mathcal{M})$.

Let $\mathcal{G}$ be the $\mathbb{S}^2$ module

$$
\mathcal{G}(m,n) = \begin{cases} 0 \ , & m < 2 \text{ or } m+n < 3 \\ H^*(M_{0,m|n}) \ , & \text{otherwise} \end{cases} \qquad (7)
$$

Let $\mathbf{D}$ be the differential operator as in Proposition 3.1. From (5) it follows that when $k \geq 3$

$$
\mathrm{ch}_t(\mathcal{G}(k,n)) = \mathfrak{m}^{(1)}_k s^{(2)}_n - t\,\mathbf{D}\,\mathfrak{m}^{(1)}_k \left(s^{(2)}_n + s^{(2)}_{n-1,1}\right) + \cdots + (-t)^n \left(\mathbf{D}^n\,\mathfrak{m}^{(1)}_k\right) s^{(2)}_{1^n} \ . \qquad (8)
$$

$\overline{M}_{0,2|n}$ are the Losev-Manin spaces and were extensively studied in [9]. Note that

$$
M_{0,2|n} \cong \left(\mathbb{C}^\times\right)^n / \mathbb{C}^\times
$$

where the quotient is taken under the diagonal action. It follows that $H^1(M_{0,2|n}) \cong V_{(1,1)} \boxtimes V_{(n-1,1)}$ (see [3, Lemma 3.3]). As before $H^k(M_{0,2|n}) \cong \wedge^k H^1(M_{0,2|n})$. Thus

$$H^k(M_{0,2|n}) \cong \begin{cases} V_{(2)} \boxtimes V_{(n-k,1^k)} & k < n \text{ even} \\ V_{(1^2)} \boxtimes V_{(n-k,1^k)} & k < n \text{ odd} \\ 0 & k \geq n. \end{cases}$$

Hence it follows that

$$\operatorname{ch}_t(\mathcal{G}(2,n)) = s_2^{(1)} s_n^{(2)} - t s_{1,1}^{(1)} s_{n-1,1}^{(2)} + \ldots = \sum_{k-0}^{n-1} (-t)^k \left(\mathbf{D}^k\, s_2^{(1)}\right) s_{n-k,1^k}^{(2)} \,. \tag{9}$$

Adding up $\operatorname{ch}_t(\mathcal{G}(k,l))$ for all k,l we get

$$\begin{aligned}\operatorname{ch}_t(\mathcal{G}) = \mathfrak{m}^{(1)} &+ \left(\mathfrak{m}^{(1)} + s_2^{(1)}\right) \sum_{n=1}^{\infty} s_n^{(2)} \\ &+ \sum_{k=1}^{\infty} (-t)^k \left(\mathbf{D}^k\, \mathfrak{m}^{(1)} \sum_{n=k}^{\infty} s_{n-k+1,1^{k-1}}^{(2)} + \mathbf{D}^k \left(\mathfrak{m}^{(1)} + s_2^{(1)}\right) \sum_{n=k+1}^{\infty} s_{n-k,1^k}^{(2)} \right).\end{aligned} \tag{10}$$

5. Cohomology of $\overline{M}_{0,m|n}$

In this section we shall determine the action of $S_m \times S_n$ on the cohomology of $\overline{M}_{0,m|n}$. To do this let us introduce the $\mathbb{S}^2$ module $\mathcal{W}$,

$$\mathcal{H}(m,n) = \begin{cases} 0\,, & m < 2 \text{ or } m+n < 3 \\ H^*(\overline{M}_{0,m|n})\,, & \text{otherwise.} \end{cases} \tag{11}$$

We shall derive a formula relating $\operatorname{ch}_t(\mathcal{G})$ (see (6)) and $\operatorname{ch}_t(\mathcal{H})$ using the partial Legendre transform.

Let X be an algebraic variety and $\emptyset \subset X_0 \subset \ldots \subset X_n = X$ a filtration on it by closed subvarieties $X_p \subset X$. Then there is a spectral sequence in cohomology with compact support (see Petersen [11, Section 1]),

$$E_1^{p,q} = H_c^{p+q}(X_p \backslash X_{p-1}) \Longrightarrow H_c^{p+q}(X).$$

The differentials of this spectral sequence are compatible with the mixed Hodge structures. Further if a finite group G acts on X and keeps each X_p invariant then $E_j^{p,q}$ has an action of G and the differentials d_j are G equivariant.

In our situation let $X = \overline{M}_{0,m|n}$ and X_p be the union of all strata of dimension at most p

$$X_p = \bigsqcup_{\substack{T \in \mathbb{T}(m,n) \\ |E(T)| = m+n-3-p}} \overline{M}(T).$$

Clearly

$$X_p \backslash X_{p-1} = \bigsqcup_{\substack{T \in \mathbb{T}(m,n) \\ |E(T)| = m+n-3-p}} M(T).$$

Thus

$$E_1^{p,q} = \bigoplus_{\substack{T \in \mathbb{T}(m,n) \\ |E(T)| = m+n-3-p}} H_c^{p+q}(M(T)). \tag{12}$$

From Proposition 4.1 and Poincaré duality it follows that the mixed Hodge structure on $H_c^i(M_{0,m|n})$ is pure of weight $2(i-m-n+3)$. This implies that $E_1^{p,q}$ has a pure Hodge structure of weight $2q$. Hence the Spectral sequence collapses in the E_2 page

$$E_2^{p,q} \cong E_\infty^{p,q}.$$

Moreover, from the fact that all the cohomology of $\overline{M}_{0,m|n}$ is algebraic it follows that

$$E_2^{p,p} \cong H^{2p}(\overline{M}_{0,m|n}) \quad \text{and} \quad E_2^{p,q} = 0 \text{ if } p \neq q \ . \tag{13}$$

Thus there is a resolution

$$H^{2p}(\overline{M}_{0,m|n}) \to \bigoplus_{\substack{T\in\mathbb{T}(m,n)\\ |E(T)|=m+n-3-p}} H_c^{2p}(M(T)) \to \cdots \to H_c^{m+n-3+p}(M_{0,m|n}) \ . \tag{14}$$

Theorem 5.1. *Let*

$$F = t^{-6}\,\mathrm{ch}_t(\mathcal{G})\Big|_{\substack{t\mapsto t^{-2}\\ p_n^{(i)}\mapsto t^{2n}p_n^{(i)}}} \ ,$$

then $h_2^{(1)} + \mathrm{ch}_t(\mathcal{H}) = \mathfrak{L}^{(1)}\left(e_2^{(1)} - F\right)$.

Remark. Note that $(e_2^{(1)} - F) \in \Lambda_*^{(2)}[\![t]\!]$. More over $\circ_{(1)}$ extends to $\Lambda^{(2)}[\![t]\!]$ in a natural way: $p_n^{(1)} \circ_{(1)} t = t^n$ and $p_n^{(2)} \circ_{(1)} t = p_n^{(2)}$. Hence $\mathfrak{L}^{(1)}$ makes sense on $\Lambda_*^{(2)}[\![t]\!]$.

Proof. As in [6, section 5.8] we shall consider (graded) $\mathbb{S}^2$ modules $\mathcal{V}$ with a further $\mathbb{Z}/2$-grading $\mathcal{V} = \mathcal{V}_{(0)} \oplus \mathcal{V}_{(1)}$. In this case define

$$\mathrm{ch}_t(\mathcal{V}) = \mathrm{ch}_t(\mathcal{V}_{(0)}) - \mathrm{ch}_t(\mathcal{V}_{(1)}) \ .$$

By Poincaré duality $H_c^k(M_{0,m|n}) \cong H^{2m+2n-6-k}(M_{0,m|n})^\vee \otimes \mathbb{C}(-m-n+3)$ where $\mathbb{C}(-\ell)$ is the ℓ-fold tensor power of the dual of the Tate Hodge structure. Thus $H_c^k(M_{0,m|n})$ has pure Hodge structure of weight $2(k-m-n+3)$.

Define the $\mathbb{Z}/2$-graded $\mathbb{S}^2$ module $\mathcal{V}$ as follows

$$\mathcal{V}(m,n) = 0 \text{ if } m < 2 \text{ or } m+n < 3 \ ,$$

otherwise

$$\mathcal{V}_{(0)}(m,n) = \bigoplus_{k=0}^{\infty} H_c^{2k}(M_{0,m|n})$$
$$\mathcal{V}_{(1)}(m,n) = \bigoplus_{k=0}^{\infty} H_c^{2k+1}(M_{0,m|n}).$$

Here we consider $H_c^k(M_{0,m|n})$ with weight grading for the mixed Hodge structure on it, that is $H_c^k(M_{0,m|n})$ is the $2(k-m-n+3)$ graded component. Then

$$F = \mathrm{ch}_t(\mathcal{V}) \ .$$

The construction $\mathbb{T}$ of Section 3.2 extends naturally to $\mathbb{Z}/2$-graded $\mathbb{S}^2$ modules (tensor product of odd and odd is even, even and even is even where as that of odd and even is odd). Proposition 3.2 generalises to the case of $\mathbb{Z}/2$-graded $\mathbb{S}^2$ modules.

If we add up all the terms of the spectral sequence (12) placing $E_1^{p,q}$ in bi-degree $2q, (p+q) \bmod 2$ we get the graded vector space $\mathbb{T}\mathcal{V}(m,n)$. The differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ gives a differential on $\mathbb{T}\mathcal{V}(m,n)$ and the resolution (14) shows that $H^*(\overline{M}_{0,m|n})$ is the homology of of the complex $(\mathbb{T}\mathcal{V}(m,n), d_1)$.

Hence $\mathrm{ch}_t(\overline{M}_{0,m|n}) = \mathrm{ch}_t(\mathbb{T}\mathcal{V}(m,n))$. This completes the proof. □

Appendix A. Calculations

Recall the $\mathbb{S}^2$ modules $\mathcal{G}$ from (7) and $\mathcal{H}$ from (11). In this section we compute the first few terms of the characteristics of $\mathcal{G}$ and $\mathcal{H}$ and list them in Table 1 and Table 2 respectively.

Formula (10) gives a recipe for calculating $\mathrm{ch}_t(\mathcal{G})$ from $\mathrm{ch}_t(\mathcal{M})$. The $\mathbb{S}$ module $\mathcal{M}$ was defined in (6). The action of S_n on $H^*(M_{0,n})$ was calculated in Getzler [6] (see Theorem 5.7). Let μ be the Möbius function, and let $R_n(t) = (1/n) \sum_{d \text{ divides } n} \mu(n/d)/t^d$. Further let κ be the linear operator on Λ which is 0 on the 0th,1st and 2nd graded components of Λ and identity on the rest. Then

$$\mathrm{ch}_t(\mathcal{M}) = \kappa\left(\frac{1+tp_1}{1-t^2}\prod_{n=1}^{\infty}(1+t^n p_n)^{R_n(t)}\right).$$

Thus using (10) we can calculate the first few terms of $\mathrm{ch}_t(\mathcal{G})$. Of course $\mathrm{ch}_t(\mathcal{G}(k,n))$ starts to be interesting when $k \geq 3$ and $n \geq 2$. In table Table 1 we list these terms for $k+n \leq 6$.

Using Theorem 5.1 we can in principle determine $\mathrm{ch}_t(\mathcal{H})$. The theorem gives a fixed point formula and first few terms of $\mathrm{ch}_t(\mathcal{H})$ can be obtained from $\mathrm{ch}_t(\mathcal{G})$ by performing several iterations. Again the terms $\mathrm{ch}_t(\mathcal{H}(k,n))$ for $n=1$ can be easily computed and are uninteresting. We list the terms corresponding to $k+n \leq 6$ and $n > 1$ in Table 2.

The calculations involving symmetric functions were done using the Maple package SF [12] by Stembridge.

(m,n)	$\mathrm{ch}_t\left(H^*(M_{0,m\mid n})\right)$
$(3,2)$	$s_3^{(1)}s_2^{(2)} - ts_{2,1}^{(1)}\left(s_2^{(2)}+s_{1^2}^{(2)}\right) + t^2\left(s_3^{(1)}+s_{2,1}^{(1)}+s_{1^3}^{(1)}\right)s_{1^2}^{(2)}$
$(3,3)$	$s_3^{(1)}s_3^{(2)} - ts_{2,1}^{(1)}\left(s_3^{(2)}+s_{2,1}^{(2)}\right) + t^2\left(s_3^{(1)}+s_{2,1}^{(1)}+s_{1^3}^{(1)}\right)\left(s_{2,1}^{(2)}+s_{1^3}^{(2)}\right) - t^3\left(s_3^{(1)}+3s_{2,1}^{(2)}+s_{1^3}^{(2)}\right)s_{1^3}^{(2)}$
$(4,2)$	$s_4^{(1)}s_2^{(2)} - t\left(s_{2^2}^{(1)}s_2^{(2)}+s_{3,1}^{(1)}\left(s_2^{(2)}+s_{1^2}^{(2)}\right)\right)+t^2\left(\left(s_4^{(1)}+s_{2^2}^{(1)}\right)s_{1^2}^{(2)}+\left(s_{3,1}^{(1)}+s_{2,1^2}^{(1)}\right)\left(s_2^{(2)}+2s_{1^2}^{(2)}\right)\right)$ $-t^3\left(s_4^{(1)}+2s_{2,2}^{(1)}+2s_{3,1}^{(1)}+2s_{2,1^2}^{(1)}+s_{1^4}^{(1)}\right)s_{1^2}^{(2)}$

TABLE 1. Equivariant Poincaré polynomial for the interior

(m,n)	$\mathrm{ch}_t\left(H^*(\overline{M}_{0,m\mid n})\right)$	Poincaré Polynomial
$(2,2)$	$(1+t^2)s_2^{(1)}s_2^{(2)}$	$1+t^2$
$(2,3)$	$(1+t^4)s_2^{(1)}s_3^{(2)}+t^2\left(s_2^{(1)}\left(s_3^{(2)}+s_{2,1}^{(2)}\right)+s_{1^2}^{(1)}s_3^{(2)}\right)$	$1+4t^2+t^4$
$(3,2)$	$(1+t^4)s_3^{(1)}s_2^{(2)}+t^2\left(s_3^{(1)}\left(2s_2^{(2)}+s_{1^2}^{(2)}\right)+s_{2,1}^{(1)}s_2^{(2)}\right)$	$1+5t^2+t^4$
$(2,4)$	$(1+t^6)s_2^{(1)}s_4^{(2)}+(t^2+t^4)\left(s_2^{(1)}\left(2s_4^{(2)}+s_{3,1}^{(2)}s_{2^2}^{(2)}\right)+s_{1^2}^{(1)}\left(s_4^{(2)}+s_{3,1}^{(2)}\right)\right)$	$1+11t^2+11t^4+t^6$
$(3,3)$	$(1+t^6)s_3^{(1)}s_3^{(2)}+(t^2+t^4)\left(s_3^{(1)}\left(3s_3^{(2)}+2s_{2,1}^{(2)}\right)+s_{2,1}^{(1)}\left(2s_3^{(2)}+s_{2,1}^{(2)}\right)\right)$	$1+15t^2+15t^4+t^6$
$(4,2)$	$(1+t^6)s_4^{(1)}s_2^{(2)}+(t^2+t^4)\left(s_4^{(1)}\left(4s_2^{(2)}+s_{1^2}^{(2)}\right)+s_{3,1}^{(1)}\left(2s_2^{(2)}+s_{1^2}^{(2)}\right)+s_{2,2}^{(1)}s_2^{(2)}\right)$	$1+16t^2+16t^4+t^6$

TABLE 2. Equivariant Poincaré polynomial of $\overline{M}_{0,m\mid n}$

References

[1] V. I. Arnol′d, *The cohomology ring of the group of dyed braids*, Mat. Zametki **5** (1969), 227–231 (Russian).

[2] J. Bergström and S. Minabe, *On the cohomology of moduli spaces of (weighted) stable rational curves*, Math. Z. **275** (2013), no. 3-4, 1095–1108.

[3] ———, *On the cohomology of the Losev-Manin moduli space*, Manuscripta Math. **144** (2014), no. 1-2, 241–252.

[4] Ö. Ceyhan, *Chow groups of the moduli spaces of weighted pointed stable curves of genus zero*, Adv. Math. **221** (2009), no. 6, 1964–1978.

[5] C. Chaudhuri, *Topological Bounds on Certain Open Subvarieties of the compactfied Moduli Space of Curves*, ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)–Northwestern University.

[6] E. Getzler, *Operads and moduli spaces of genus* 0 *Riemann surfaces*, The moduli space of curves (Texel Island, 1994), Progr. Math., vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 199–230.

[7] E. Getzler and M. M. Kapranov, *Modular operads*, Compositio Math. **110** (1998), no. 1, 65–126.

[8] B. Hassett, *Moduli spaces of weighted pointed stable curves*, Adv. Math. **173** (2003), no. 2, 316–352.

[9] A. Losev and Y. Manin, *New moduli spaces of pointed curves and pencils of flat connections*, Michigan Math. J. **48** (2000), 443–472. Dedicated to William Fulton on the occasion of his 60th birthday.

[10] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1995. With contributions by A. Zelevinsky; Oxford Science Publications.

[11] D. Petersen, *The structure of the tautological ring in genus one* (2012), available at `http://arxiv.org/abs/1205.1586`.

[12] J. Stembridge, *The SF package*, available at `http://www.math.lsa.umich.edu/~jrs/maple.html`. A Maple package to do symbolic calculations with symmetric functions.